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A lattice model with an infinite number of phase transitions

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Abstract. A lattice model first proposed by Ginibre and a related model are considered in the classical or equivalent neighbour limit. It is found that there exists an infinite number of phase transitions in the appropriate field space.

1. Introduction

Some time ago J Ginibre (1971, unpublished) proposed a lattice model with Hamiltonian

$$\mathscr{H} = -\sum_{1\leq i<j\leq N} J_{ij}n_in_j + \lambda J\sum_{i=1}^{N} n_i^2 - H\sum_{i=1}^{N} n_i \tag{1.1}$$

where $n_i = 0, 1, 2, \ldots$, as a model which might have an infinite number of phase transitions. Here we investigate the classical or equivalent neighbour version of this model with

$$J_{ij} = J/N > 0 \qquad \text{for all } i, j \tag{1.2}$$

and $\lambda > \frac{1}{2}$ so that the thermodynamic limit exists.

A somewhat more tractable version of the model which we will treat first and refer to as model I allows the site variables n_i to take both positive and negative integral values. Ginibre's original model to which we will refer as model II is discussed in § 3 and we conclude with several conjectures and comments in § 4.

In order to investigate the phase diagrams of these models, we found it necessary to work in the (T, λ, H) field space (Griffiths and Wheeler 1970) where H is the field conjugate to the order parameter

$$m(T, \lambda, H) = \langle n_i \rangle \tag{1.3}$$

and $\langle \ldots \rangle$ denotes thermal average. It is only in this space that rich critical behaviour of the models becomes manifest. For example, in (T, λ, H) space the occurrence of an infinite number of phase transitions for model I is virtually guaranteed for any translationally invariant symmetric interaction J_{ij}. This is easily seen by noting that the free energy $f(\beta, \lambda, H)$ in the thermodynamic limit, given by

$$-\beta f(\beta, \lambda, H) = \lim_{N\to\infty} N^{-1} \ln\left(\sum_{\{n_i\}} \exp(-\beta\mathscr{H})\right) \tag{1.4}$$

satisfies

$$f(\beta, \lambda, H) = f\left(\beta, \lambda, H + \left(2\lambda J - \sum_{j(\neq i)} J_{ij}\right)\right) + H + \lambda J - \tfrac{1}{2}\sum_{j(\neq i)} J_{ij} \tag{1.5}$$

which, for any translationally invariant symmetric J_{ij}, can be shown by changing variables n_i to $n_i - 1$. Equation (1.5) shows that apart from additive analytic terms, the free energy is a periodic function of H so that if there is a phase transition for a certain value of $H = H_c$, then there will be an infinite number of phase transitions occurring when $H = H_c + k(2\lambda J - \Sigma_{j(\neq i)} J_{ij})$, $k = 0, \pm 1, \pm 2, \ldots$.

The free energy of model II is of course not strictly periodic but, as we will see, the calculation presented in § 3 indicates rather strongly that whenever model I has an infinite number of (or equivalently one) phase transitions then so does model II.

We now turn to the equivalent neighbour version of our models.

2. Model I: $n_i = 0, \pm 1, \pm 2, \ldots$

Using the elementary identity

$$e^{A^2/2} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-z^2/2} e^{Az}\, dz \tag{2.1}$$

with

$$A = \sqrt{(\beta J/N)} \sum_{i=1}^{N} n_i,$$

the partition function for the equivalent neighbour model may be written as

$$\begin{aligned}
\sum_{\{n_i\}} e^{-\beta \mathcal{H}} &= \sum_{\{n_i\}} \exp\left[\beta J/2N\left(\sum_{i=1}^{N} n_i\right)^2 - \beta J/2N \sum_{i=1}^{N} n_i^2 - \beta\lambda J \sum_{i=1}^{N} n_i^2 + \beta H \sum_{i=1}^{N} n_i\right] \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-z^2/2} \sum_{\{n_i\}} \exp\left(-\beta J[\lambda + (1/2N)] \sum_{i=1}^{N} n_i^2\right. \\
&\quad \left. + [\beta H + z\sqrt{(\beta J/N)}] \sum_{i=1}^{N} n_i\right) dz \\
&= (\beta JN/2\pi)^{1/2} \int_{-\infty}^{\infty} \left(e^{-\beta J y^2/2} \sum_{n=-\infty}^{\infty} \exp\{-\beta J[\lambda + (1/2N)]n^2\right. \\
&\quad \left. + (\beta H + \beta J y)n\}\right)^N dy
\end{aligned} \tag{2.2}$$

where in the last step we have made the change of variables $z = y\sqrt{(\beta JN)}$. Making the further change of variables

$$\beta H + \beta J y = 2bx, \qquad b = \beta J\lambda \tag{2.3}$$

and applying Laplace's method to the integral (2.2) we then obtain from (1.4) the expression for the limiting free energy per site (cf Thompson 1972):

$$-\beta f(\beta, \lambda, H) = \max_{-\infty < x < \infty} [g_1(x, b) - 2b(\lambda - \tfrac{1}{2})(x - \bar{H})^2] + (\lambda - \tfrac{1}{2})\beta J \bar{H}^2 \tag{2.4}$$

with the reduced field $\bar{H}$ defined by

$$\bar{H} = H/2(\lambda - \tfrac{1}{2})J \tag{2.5}$$

and

$$g_1(x, b) = \ln\left(\sum_{n=-\infty}^{\infty} e^{-b(n-x)^2}\right). \tag{2.6}$$

It will be noted in terms of $\bar{H}$, equation (1.5) for model I becomes

$$f(\beta, \lambda, \bar{H}) = f(\beta, \lambda, \bar{H}+1) - (\lambda - \tfrac{1}{2})J(1+2\bar{H}) \tag{2.7}$$

and for the order parameter (1.3)

$$m(\beta, \lambda, \bar{H}+1) = m(\beta, \lambda, \bar{H}) + 1. \tag{2.8}$$

Further, if $x^*(\beta, \lambda, \bar{H})$ is the maximizing x in (2.4), it is easily shown that

$$m(\beta, \lambda, \bar{H}) = 2\lambda(x^*(\beta, \lambda, \bar{H}) - \bar{H}) + \bar{H}. \tag{2.9}$$

The exponential of the function $g_1(x, b)$, which is periodic and even in x, can be expressed in terms of a theta function with imaginary argument (see, for example, Whittaker and Watson 1965). Using the appropriate Jacobi infinite product representation of the theta function we obtain the expression

$$g_1(x, b) = -bx^2 + \sum_{n=1}^{\infty} [\ln(1 - e^{-2nb}) + \ln(1 + e^{-2(n-\frac{1}{2}+x)b}) + \ln(1 + e^{-2(n-\frac{1}{2}-x)b})]. \tag{2.10}$$

Numerically, $g_1(x, b)$ has a maximum at $x = 0$ and decreases monotonically as x increases until $x = \frac{1}{2}$ where it reaches a minimum. Consequently, from (2.4), when $\bar{H} = \frac{1}{2}$ there are two maximizing values for x^*, corresponding to a jump discontinuity in the order parameter $m(\beta, \lambda, \bar{H})$ (as a function of $\bar{H}$), provided

$$\left.\frac{d^2 g_1}{dx^2}\right|_{x=1/2} > 4b(\lambda - \tfrac{1}{2}). \tag{2.11}$$

From the periodic properties (2.7) and (2.8) we therefore have an infinite number of phase transitions at

$$\bar{H} = \tfrac{1}{2} \pm k, \qquad k = 0, 1, 2, \ldots \tag{2.12}$$

provided

$$T < T_c(\lambda) \tag{2.13}$$

where from (2.3) and (2.11) the critical temperature $T_c(\lambda)$ is determined parametrically through the equations

$$\frac{kT_c(\lambda)}{J} = \frac{\lambda}{b} \tag{2.14}$$

and, from (2.10),

$$2\left(\lambda - \frac{1}{2}\right) = \frac{1}{2b} \left.\frac{d^2 g_1}{dx^2}\right|_{x=1/2} = b \sum_{n=1}^{\infty} (\cosh nb)^{-2} + \frac{b}{2} - 1. \tag{2.15}$$

Actual values for $T_c(\lambda)$ determined numerically are shown in figure 1.

Phase diagrams for model I are shown in figures 2(a) and 2(b) for $\lambda = 1$. For different values of $\lambda\,(>\frac{1}{2})$ the overall features remain the same. In figure 2(a), the coexistence lines at $\bar{H} = \pm(k - \frac{1}{2})$, $k = 1, 2, \ldots$ are straight lines terminating at the critical points $c_{\pm k}$. The values of m at these critical points are $\pm(k - \frac{1}{2})$ respectively and due to the periodic nature of the free energy, the critical behaviour is the same at all critical points. Since the model is essentially a mean field model all critical exponents are classical, or in other words, the coexistence curves in the m–T plane are parabolic in the neighbourhood of the critical points.

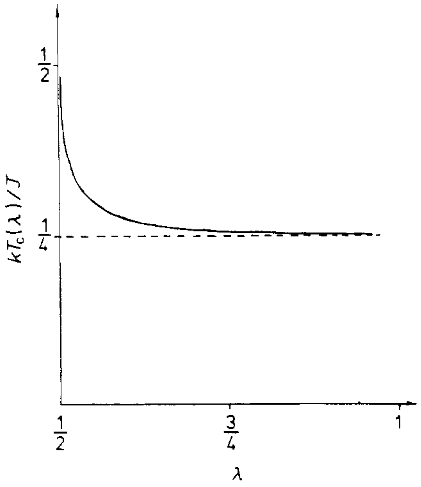

Figure 1. Numerical values of $T_c(\lambda)$. As $\lambda \to \infty$, $kT_c(\lambda)/J$ approaches to $\frac{1}{4}$ exponentially.

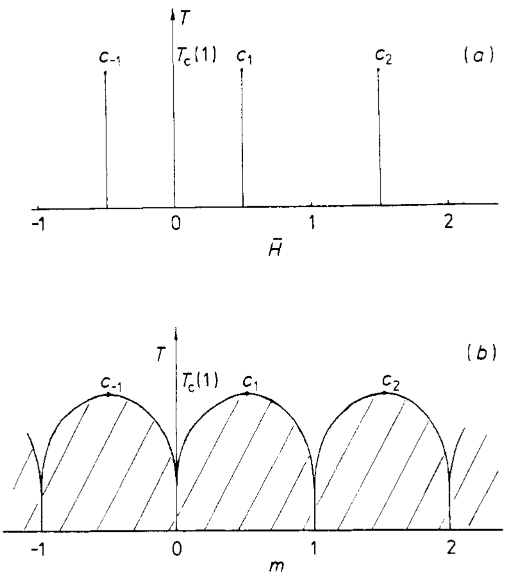

Figure 2. Phase diagrams of model I in (*a*) T–H plane and in (*b*) T–m plane for $\lambda = 1$. $c_{\pm k}$, $k = 1, 2, \ldots$, are the critical points whose critical behaviours are all identical. The shaded regions in (*b*) are the two phase regions. Here $kT_c(1)/J = 0{\cdot}2507$.

3. Model II: $n_i = 0, 1, 2, \ldots$

For the original Ginibre model we do not have a periodicity property such as (1.5) for model I. Nevertheless, in the equivalent neighbour case at least, model II also has an infinite number of phase transitions.

The derivation of the free energy for model II parallels that for model I. The final expression is the same as (2.4) with $g_1(x, b)$ replaced by $g_2(x, b)$ defined by

$$g_2(x, b) = \ln\left(\sum_{n=0}^{\infty} \mathrm{e}^{-b(n-x)^2}\right). \tag{3.1}$$

In this case the location of the singularities must be determined numerically. Some general features can, however, be seen in some limiting cases. Firstly, if x is large and positive

$$g_2(x, b) = g_1(x, b) + \mathrm{O}(\mathrm{e}^{-b(1+x)^2}). \tag{3.2}$$

Therefore, if $\bar{H}$ is large, then x^* and m are large and we can expect the behaviour of the free energy to be similar to that of model I in this limit. Secondly, when $b \to \infty$, that is, if either $\lambda \to \infty$ or $T \to 0$, $g_2(x, b)$ becomes asymptotically

$$g_2(x, b) \sim \begin{cases} g_1(x, b) & x > 0 \\ -bx^2 & x < 0. \end{cases} \tag{3.3}$$

From this we can deduce that for $\bar{H} > 0$ (or $m > 0$), the thermodynamic behaviour of the two models is the same in the low temperature limit for all λ, or in the large λ limit for all T, while for $\bar{H} < 0$ (or $m < 0$), there is no phase transition at all for model II in the above mentioned limits. Consequently the phase diagrams of model II in these limits are the same as that of figure 2 for $\bar{H} > 0$ and $m > 0$ and no phase transitions for $\bar{H} < 0$ and $m < 0$.

Actual shapes of the phase diagrams for model II have been determined numerically. In figure 3 we show the case for $\lambda \to \infty$ where the coexistence lines in the T–$\bar{H}$ plane are all straight lines. For finite λ they deviate from straight lines and bend towards the left, the deviation being greatest for small $\bar{H}$ and exponentially vanishing as $\bar{H}$ increases. From a numerical point of view all phase diagrams are essentially the same for $\lambda \gtrsim 1$.

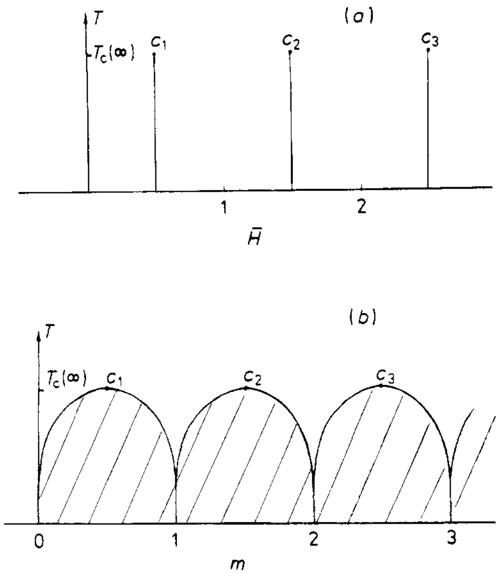

Figure 3. Phase diagrams of model II for $\lambda = \infty$ in (a) T–H plane and in (b) T–m plane. $kT_c(\infty)/J = 0{\cdot}25$. Within this scale, the figures do not change for all λ greater than 1.

The differences between the phase diagrams for small $\bar{H}$ or m compared with those for large $\bar{H}$ or m become prominent when λ approaches $\frac{1}{2}$. As a typical case we show the situation in figure 4 for $\lambda = 0{\cdot}51$. The coexistence line near $\bar{H} = 2{\cdot}5$ is not strictly straight; at the critical point c_3 the critical field is $\bar{H}_c = 2{\cdot}4999929...$ (rather than $2{\cdot}5$).

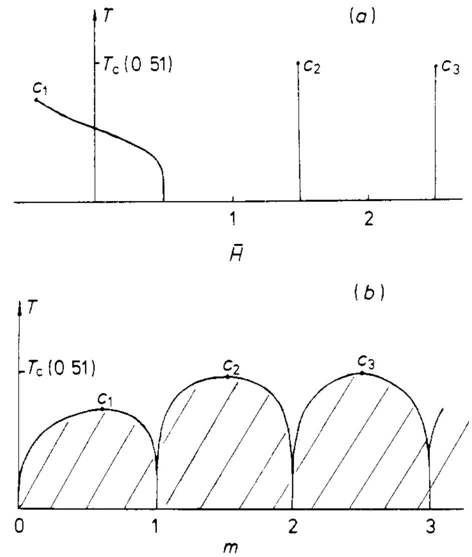

Figure 4. Phase diagrams of model II for $\lambda = 0{\cdot}51$ in (*a*) T–H plane and in (*b*) T–m plane. $kT_c(0{\cdot}51)/J = 0{\cdot}3765$.

We note also that the coexistence line below the critical point c_1 in this case crosses the T axis so that in zero field there is a first order phase transition (at $kT/J = 0{\cdot}2004...$). Also the coexistence lines in the T–H plane near the critical points are not parallel to the T axis so that the specific heat of model II is a strongly diverging quantity, like the susceptibility, at each of the critical points (cf Griffiths and Wheeler 1970), in this case with classical exponents $\alpha = \alpha' = \gamma = \gamma' = 1$ if one approaches a critical point asymptotically parallel to its coexistence line and with $\alpha = \alpha' = 1 - (1/\delta) = \frac{2}{3}$ otherwise. In addition, as λ approaches closer to $\frac{1}{2}$ more lines cross the T axis resulting in a multiplicity of first order phase transition points in zero field. This is in contrast to model I whose free energy is always analytic in zero field.

4. Discussion

In this article we have considered two models and shown that in the equivalent neighbour case, or classical limit, both have an infinite number of phase transitions, with classical critical exponents at all critical points.

For more realistic short ranged interactions J_{ij}, e.g. nearest neighbour interactions only, we expect the qualitative features of the phase diagrams (figures 2, 3 and 4), and the fact that there is an infinite number of phase transitions for both models, to remain valid, at least for lattice dimensionality greater than, and possibly equal to, two.

Furthermore, if in general, we let

$$\bar{H}=H\Big(2\lambda J-\sum_{j(\neq i)}J_{ij}\Big)^{-1}=\tfrac{1}{2}+k, \qquad k=0,\pm1,\pm2,\ldots, \tag{4.1}$$

the Hamiltonian (1.1) for model I can be rewritten in the form

$$\mathscr{H}=-\sum_{i<j}J_{ij}\sigma_i\sigma_j+\lambda J\sum_i\sigma_i^2-\tfrac{1}{2}N\Big(\frac{1}{2}+k\Big)^2\Big(2\lambda J-\sum_{j(\neq i)}J_{ij}\Big) \tag{4.2}$$

where

$$\sigma_i=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\pm\tfrac{5}{2},\ldots.$$

It follows that in the limit $\lambda\to\infty$ and $H\to\infty$ such that $\bar{H}=\frac{1}{2}$, the free energy becomes

$$f=f_{\mathrm{I}}+\tfrac{1}{8}\sum_{j(\neq i)}J_{ij} \tag{4.3}$$

where f_{I} is the free energy of the zero field Ising model with Hamiltonian

$$\mathscr{H}_{\mathrm{I}}=-\sum_{i<j}J_{ij}\sigma_i\sigma_j \qquad (\sigma_i=\pm\tfrac{1}{2}). \tag{4.4}$$

In other words, in the limit $\lambda\to\infty$, the critical point c_1 of figure 2 becomes that of the Ising model. Hence, if we assume that the critical behaviour of model I falls in the same universality class (Griffiths 1970) for all $\lambda>\frac{1}{2}$, then we can expect that the critical exponents associated with the critical points $c_{\pm1}$, $c_{\pm2}$, ... to be all of Ising type. Exactly the same observations can be made for model II. It is to be noted also that when expressed in the form (4.2) the anomalous feature of model I, being analytic in zero field, disappears and the model resembles more closely a spin model, which may be thought of as a discrete infinite spin version of the Ising model.

Although we have no proofs of our conjectures at this stage, it would seem that it should be at least possible to prove that if an Ising model has phase transition, then the corresponding models I or II have an infinite number of phase transitions.

Acknowledgment

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